On Performance Limitations of Aperture Coupling Between Rectangular Waveguides

CHANG-HONG LIANG AND DAVID K. CHENG, FELLOW, IEEE

Abstract—**The conditions for and expressions of the best coupling flatness and optimum directivity obtainable over a given bandwidth from broad-wall aperture coupling between rectangular waveguides are derived. An inequality representing a constraining relationship among coupling flatness, directivity, and bandwidth is established.**

I. Introduction

DESPITE THE importance of wide-band waveguide directional couplers using discrete apertures, few studies have been made on the conditions for optimum single-aperture performance from the synthesis viewpoint. In this paper, we first obtain a formula for the best coupling flatness attainable over a prescribed bandwidth for a small aperture of an arbitrary shape in the common broad wall between two rectangular waveguides. The conditions for achieving the optimum coupling flatness are derived and applied to both circular and T-shaped slot-pair apertures. Next, we examine the minimum coupling directivity over the bandwidth and find the requirements for maximizing this minimum directivity. Thirdly, we formulate an inequality which shows the interrelationship among the coupling flatness coefficient, the minimum coupling directivity, and the relative waveguide bandwidth.

II. Optimum Coupling Flatness

Consider two rectangular waveguides with a common broad wall in which a single aperture is cut, as shown in Fig. 1. The dimensions of the aperture in terms of wavelength are such that Bethe's small-hole coupling theory applies [1]. Assuming negligible wall thickness, the normalized amplitude of the forward scattering wave for the dominant TE_{10} mode is [2]

$$A=\frac{j2\pi}{ab\lambda_g}\left\{\left[M_x-\left(\frac{\lambda_g}{\lambda}\right)^2P_y\right]\sin^2\left(\frac{\pi x}{a}\right)+M_z\left(\frac{\lambda_g}{2a}\right)^2\cos^2\left(\frac{\pi x}{a}\right)\right\} \quad (1)$$

and the normalized amplitude of the backward scattering wave is

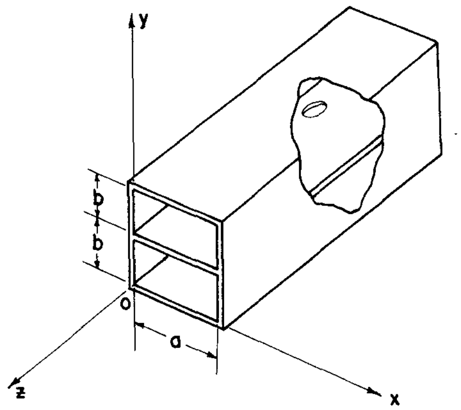

Fig. 1. Rectangular waveguide with broad-wall aperture coupling.

$$B=-\frac{j2\pi}{ab\lambda_g}\left\{\left[M_x+\left(\frac{\lambda_g}{\lambda}\right)^2P_y\right]\sin^2\left(\frac{\pi x}{a}\right)-M_z\left(\frac{\lambda_g}{2a}\right)^2\cos^2\left(\frac{\pi x}{a}\right)\right\} \quad (2)$$

where P_y is the electric polarizability, and M_x and M_z are the x- and z-components of the magnetic polarizability of the small aperture, respectively. It is convenient to define a new variable

$$u=\frac{2a}{\lambda_g} \quad (3)$$

and to indicate the transverse position of the aperture by X

$$X=\sin^2\left(\frac{\pi x}{a}\right). \quad (4)$$

Then A in (1) can be written as

$$A=\frac{j\pi}{a^2b}f(u) \quad (5)$$

where

$$f(u)=ku+\frac{h}{u} \quad (6)$$

$$k=(M_x-P_y)X \quad (7)$$

$$h=M_z(1-X)-P_yX. \quad (8)$$

Manuscript received September 3, 1981; revised December 1, 1981.

The authors are with the Electrical and Computer Engineering Department, Syracuse University, Syracuse, NY 13210.

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The nature of the frequency dependence of A is now succinctly contained in the function $f(u)$ in (6), and we need only to examine $f(u)$ in regard to the variations of coupling coefficient over a given frequency bandwidth. The parameters k and h depend on the polarizabilities as well as on the transverse position of the coupling aperture but are independent of frequency.

We wish to determine the relative values of k and h which will minimize the variation of the function $f(u)$ over a given bandwidth $[u_1, u_2]$ and to find a measure of this optimum coupling flatness. Writing

$$\xi = \frac{k}{h} \tag{9}$$

we have

$$f(u,\xi) = k\left(u + \frac{1}{\xi u}\right) \tag{10}$$

which has a *single* minimum f_m at $u = 1/\sqrt{\xi}$

$$f_m(\xi) = \frac{2k}{\sqrt{\xi}}. \tag{11}$$

At the ends, u_1 and u_2, of the prescribed frequency band, the values of $f(u_1,\xi)$ and $f(u_2,\xi)$ are larger than $f_m(\xi)$, all of which depend on the value of ξ. The extent of the variation of the ratio $f(u,\xi)/f_m(\xi)$ over the range $[u_1, u_2]$ is a measure of coupling flatness. For optimum flatness we make the values of this ratio at the two ends equal; that is

$$f(u_1,\xi) = f(u_2,\xi) = f_M(\xi) \tag{12}$$

where f_M denotes the maximum value of the function $f(\xi)$. Equation (12), in conjunction with (10), yields the condition for optimum flatness

$$\xi_0 = \frac{1}{u_1 u_2}. \tag{13}$$

With (13), we have, from (10) and (11)

$$f_M(\xi_0) = k(u_1 + u_2) \tag{14}$$

and

$$f_m(\xi_0) = 2k\sqrt{u_1 u_2}. \tag{15}$$

It is convenient to refer the variation of coupling to the geometric mean of $f_M(\xi_0)$ and $f_m(\xi_0)$ and define the optimum coupling flatness in decibels by

$$(\Delta C)_0 = \pm 20\log_{10}\frac{f_M(\xi_0)}{\sqrt{f_M(\xi_0)f_m(\xi_0)}} = \pm 10\log_{10}\left[\frac{f_M(\xi_0)}{f_m(\xi_0)}\right] = \pm 10\log_{10}\frac{u_1+u_2}{2\sqrt{u_1u_2}}. \tag{16}$$

Equation (16) is independent of the shape of the aperture as long as the small-hole theory holds. For full-band operation of the dominant mode in a rectangular waveguide $u_1 \approx 0.75$ and $u_2 \approx 1.62$, we obtain

$$(\Delta C)_0 = \pm 0.314\,\text{dB} \tag{17}$$

which is the minimum obtainable coupling variation.

III. Conditions for Optimum Coupling Flatness

Although (16) indicates that the optimum coupling flatness $(\Delta C)_0$ is not a function of aperture shape, the condition, as given in (13), for achieving this result depends closely on the shape and the transverse position of the aperture. We now consider two important special cases; namely, a circular aperture and a T-shaped slot pair.

For a small circular hole of diameter d in the broad wall of a rectangular waveguide, as shown in Fig. 2, we have

$$M_x = M_z = M = \frac{d^3}{6} \tag{18}$$

and

$$P_y = \frac{M}{2}. \tag{19}$$

From (7) and (8), we find $k = MX/2$ and $h = M(1-3X/2)$ which, in conjunction with (9) and (13), lead to the requirement

$$X = \frac{2}{3 + u_1 u_2} \tag{20}$$

or

$$\frac{x}{a} = \frac{1}{\pi}\sin^{-1}\sqrt{\frac{2}{3+u_1u_2}}. \tag{21}$$

Using $u_1 = 0.75$ and $u_2 = 1.62$, we obtain

$$\frac{x}{a} = 0.242. \tag{22}$$

This result confirms with the findings of Cohn *et al.* [2], who pointed out that for optimum coupling flatness, x/a should be slightly less than 0.25.

If a T-shaped slot pair is used as in Fig. 3, the parameter X defined in (4) is unity in association with M_x and P_y of the transverse slot and is zero corresponding to M_z of the longitudinal slot. Equations (7) and (8) become

$$k = M_x - P_y \tag{23}$$

$$h = M_z - P_y. \tag{24}$$

In practice, the slots are very narrow and P_y may be neglected. With this approximation the condition for optimum coupling flatness (13) reduces to

$$\frac{M_x}{M_z} = \frac{1}{u_1 u_2}. \tag{25}$$

For $u_1 = 0.75$ and $u_2 = 1.62$, (25) gives

$$M_z = 1.215 M_x. \tag{26}$$

This result, which requires M_z to be slightly larger than M_x, has been noted experimentally by Riblet and Saad [3]. If

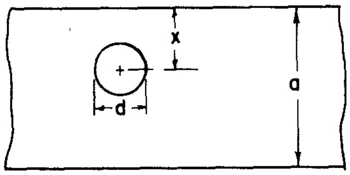

Fig. 2. Single circular aperture in broad-wall.

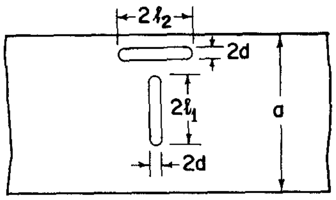

Fig. 3. T-shaped slot pair in broad-wall.

the slots have the same width, l_2 should then be slightly longer than l_1 for coupling flatness.

IV. Optimum Directivity Limit

Another important problem for directional couplers is that of maximizing the minimum directivity over the frequency band $[u_1, u_2]$. Directivity D is defined as the ratio of the forward power to the backward power

$$D = 20\log_{10}\left|\frac{A}{B}\right| \tag{27}$$

which is frequency-dependent. By using $f(u)$ defined in (6) and introducing

$$s = 2M_x X \tag{28}$$

we can simplify the expressions of A and B given in (1) and (2) and write (27) as

$$D = 20\log_{10}\left|\frac{f(u)}{su - f(u)}\right|. \tag{29}$$

Similar to the argument used in arriving at the condition for optimum coupling flatness, it can be shown that minimum directivities occur at both ends of the frequency band and the minimum over the band is maximized when the directivities at u_1 and u_2 are made equal. It is convenient to invert the fraction within the absolute signs in (29)

$$\left|\frac{su - f(u)}{f(u)}\right| = \left|\frac{s}{k + \frac{h}{u^2}} - 1\right|. \tag{30}$$

We require

$$\frac{s}{k + \frac{h}{u_2^2}} - 1 = 1 - \frac{s}{k + \frac{h}{u_1^2}} \tag{31}$$

and a minimum s. From (7) and (28) we know that

$$s = 2(k + P_x X) \geqslant 2k. \tag{32}$$

Writing $\xi = k/h$ as in (9), (31), and (32) yield the following conditions for maximizing the minimum directivity D_m over the band $[u_1, u_2]$

$$\xi_0 = \frac{1}{u_1 u_2} \tag{33a}$$

$$s_0 = 2k. \tag{33b}$$

When (33a) and (33b) are satisfied, we find the maximized minimum (optimum) directivity at both ends of the frequency band to be

$$(D_m)_0 = 20\log_{10}\left(\frac{u_2 + u_1}{u_2 - u_1}\right). \tag{34}$$

For $u_1 = 0.75$ and $u_2 = 1.62$, (34) works out to be

$$(D_m)_0 = 8.705\,\text{dB}. \tag{35}$$

We can now draw two important conclusions. First, we see from (33a) and (13) that the same k/h ratio gives an optimum coupling flatness $(\Delta C)_0$, and a maximum value of the minimum directivity $(D_m)_0$. This fact was also noted experimentally by Riblet and Saad [3]. Second, (33b) requires $P_y = 0$; consequently, a circular hole having a nonvanishing P_y cannot reach an optimum directivity limit. In the following section we examine what directivity a single circular aperture can provide.

V. Directivity Limit for Single Circular Aperture

For a circular aperture, (18) and (19) hold, and the combination of (7) and (28) gives

$$s = 4k. \tag{36}$$

Substitution of (36) in (31) yields

$$\xi = \frac{k}{h} = \frac{\sqrt{(u_1^2 + u_2^2)^2 + 12u_1^3 u_2^2} - (u_1^2 + u_2^2)}{6u_1^2 u_2^2}. \tag{37}$$

The corresponding directivity at both ends of the frequency band is

$$D_m = 20\log_{10}\left[\frac{1}{1 - \left(\frac{4\xi u_1^2}{\xi u_1^2 + 1}\right)}\right] = 20\log_{10}\left[\frac{1}{\left(\frac{4\xi u_2^2}{\xi u_2^2 + 1}\right) - 1}\right]. \tag{38}$$

With $u_1 = 0.75$ and $u_2 = 1.62$, (37) and (38) become, respectively

$$\xi = \frac{k}{h} = 0.236 \tag{39}$$

and

$$D_m = 5.501\,\text{dB} \tag{40}$$

which is seen to be less than $(D_m)_0$ in (35).

Use of (18) and (19) in (7) and (8) gives

$$\xi = \frac{X}{2-3X} \tag{41}$$

from which the transverse position of the circular hole can be found

$$X = \frac{2\xi}{1+3\xi} = 0.277 \tag{42}$$

and

$$\frac{x}{a} = \frac{1}{\pi}\sin^{-1}\sqrt{0.277} = 0.176. \tag{43}$$

We note that this position is different from that in (22) for optimum coupling flatness. In general, optimum coupling flatness is a more important consideration inasmuch as a higher directivity can be obtained by using an array of holes.

In some cases, it may be desired that the directivity does not become negative ($A \geqslant B$ in (27)) on $[u_1, u_2]$. The condition for this constraint with a single circular aperture can be found by using (18) and (19) in (29)

$$D = 20\log_{10}\left|\frac{u^2X+(2-3X)}{3u^2X-(2-3X)}\right|. \tag{44}$$

In the range of our interest, both the numerator and the denominator in (44) are positive. To make $D \geqslant 0$, we require

$$u^2X+(2-3X) \geqslant 3u^2X-(2-3X) \tag{45}$$

over $[u_1, u_2]$, or

$$P \leqslant \frac{2}{3+u_2^2}$$

which, for $u_2 = 1.62$, is equivalent to

$$\frac{x}{a} \leqslant 0.203. \tag{46}$$

This was the value given in [4]. In view of (22), this value does not yield an optimum coupling flatness.

VI. Coupling Inequality

In the above, we have studied the conditions for optimum coupling flatness and optimum directivity in the design of directional couplers using a single aperture (including a slot pair) in the common broad wall between two rectangular waveguides. In general, these conditions are not satisfied simultaneously and it is interesting to examine what the relation is among the three factors: coupling flatness, minimum directivity, and waveguide bandwidth. To this end, we define a coupling flatness coefficient $\Delta\mathcal{C}$, a minimum directivity coefficient $\mathcal{D}_m$, and a relative waveguide bandwidth $\mathcal{W}_\lambda$ as follows:

$$\Delta C = \pm 10\log_{10}(\Delta\mathcal{C}) \tag{47}$$

$$D_m = 20\log_{10}\mathcal{D}_m \tag{48}$$

$$\mathcal{W}_\lambda = \frac{\lambda_{g1}-\lambda_{g2}}{\lambda_{g0}} \tag{49}$$

where λ_{g1} and λ_{g2} correspond to $2a/u_1$ and $2a/u_2$, respectively, and λ_{g0} is the midband guide wavelength

$$\lambda_{g0} = \frac{2\lambda_{g1}\lambda_{g2}}{\lambda_{g1}+\lambda_{g2}} = \frac{4a}{u_1+u_2}. \tag{50}$$

With (50), $\mathcal{W}_\lambda$ in (49) can be written as

$$\mathcal{W}_\lambda = \frac{u_2^2-u_1^2}{2u_1u_2}. \tag{51}$$

Compared to the optimum values, $\Delta C \geqslant (\Delta C)_0$ and $D_m \leqslant (D_m)_0$. Consequently, from (16) and (34), we can write

$$\Delta\mathcal{C} \geqslant \frac{u_1+u_2}{2\sqrt{u_1u_2}} \tag{52}$$

and

$$\mathcal{D}_m \leqslant \frac{u_2+u_1}{u_2-u_1}. \tag{53}$$

Combination of (51), (52), and (53) yields the following inequality:

$$\frac{\mathcal{W}_\lambda\mathcal{D}_m}{(\Delta\mathcal{C})^2} \leqslant 2. \tag{54}$$

When the conditions for optimum coupling flatness and optimum directivity are simultaneously satisfied over a specified frequency band

$$\mathcal{W}_\lambda(\mathcal{D}_m)_0 = 2(\Delta\mathcal{C})_0^2. \tag{55}$$

It is obvious from (54) that a wider bandwidth results in either a poorer coupling flatness or a lower minimum directivity, or both.

VII. Conclusion

It is not accidental that there exist performance limitations for directional couplers with a single aperture between contiguous waveguides inasmuch as the coupling properties of the electric and magnetic fields are frequency- and position-dependent. In this paper, we have examined the conditions for and the expressions of the best coupling flatness and optimum directivity obtainable over a given bandwidth. We have also formulated an inequality which shows the upper bound of a relationship among the coupling flatness coefficient, the minimum coupling directivity, and the relative waveguide bandwidth.

Our work is based on Bethe's small-hole theory and no consideration has been given to the finite thickness of the waveguide wall. Preliminary analytical investigation has indicated that larger apertures with a finite thickness result in a poorer coupling flatness. More study in this respect is needed in the light of recent work by MacDonald [5] and Levy [6].

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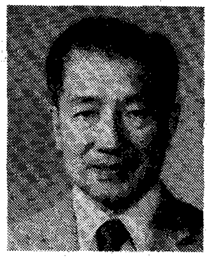
Chang-Hong Liang was born in 1943 in Shanghai, China. He graduated from the Department of Electromagnetic Engineering, Northwest Telecommunication Engineering Institute, Xian, People's Republic of China, in July 1965. After graduation he remained at the Institute as a Graduate Assistant and then as a Lecturer. He has done research work in the areas of ferrite digital phase shifter, phased antenna arrays, and microwave directional couplers. In August 1980 he came to the United States as a Visiting Scholar, under the guidance of Professor D. K. Cheng of Syracuse University.



David K. Cheng (S'44–A'48–SM'50–F'60) obtained his B.S. degree in electrical engineering from the National Chiao-Tung University in China. He went to the United States in 1943 for postgraduate studies at Harvard University where he received the S.M and Sc.D. degrees in 1944 and 1946, respectively. While at Harvard, he was a Charles Storrow Scholar and a Gordon McKay Scholar.

Dr. Cheng was associated with the Communications Laboratory of the USAF Cambridge Research Laboratories from 1946 to 1948. In 1948 he joined the faculty of Syracuse University where he is a Professor of Electrical Engineering. He was honored by Syracuse University as a Centennial Professor in 1970, and received the Chancellor's Citation in 1981 for exceptional academic achievement.

Dr. Cheng has published one book, *Analysis of Linear Systems* (Addison-Wesley), in 1959, and numerous journal articles on electromagnetic theory, antennas and arrays, and communication and signal-processing systems. Another book, *Field and Wave Electromagnetics*, is due to be published in 1982. He was a member of the IEEE Publications Board for 1968–1970; a member of the Administrative Committee of IEEE Antennas and Propagation Society, 1972–1974; the Consulting Editor of an Electrical Engineering Monograph series for the Intext Educational Publishers, 1969–1972; and a Consulting Editor for Electrical Science books published by the Addison-Wesley Publishing Co., 1961–1978.

In 1960–1961 Dr. Cheng was awarded a Guggenheim Fellowship to study and travel in Europe. For the 1975–1976 year he served as a Liaison Scientist of the London Branch of the Office of Naval Research, on leave from Syracuse University. The IEEE Antennas and Propagation Society appointed him as the European Lecturer under its Distinguished Lecturer Program for the same period.

An Alternative Theory of Optical Waveguides with Radial Inhomogeneities

ANDREAS TONNING, SENIOR MEMBER, IEEE

***Abstract*—The field equations are solved for an inhomogeneous dielectric cylinder with azimuthal symmetry. The solutions are shown to satisfy particular orthogonality relations and allow derivation of simple, generally valid expressions for dispersion relation, power flow, energy density, and group delay. A method for numerical solution of the equations, the modified staircase method, is proposed. It is shown that it leads to expressions similar to those of the Wentzel-Kramer-Brillouin (WKB) method, but, unlike the latter, is valid for the lowest order guided modes. The method has been tested in a computer program.**

Manuscript received October 2, 1981; revised November 11, 1981. This work was supported by the Royal Norwegian Council for Scientific and Industrial Research.

The author is with the Department of Electrical Engineering, Norwegian Institute of Technology (NTH), Trondheim, N-7034, Norway.

I. INTRODUCTION

THE PRESENT paper presents a theory of wave propagation on an inhomogeneous optical waveguide of cylindrical symmetry. The discussion is based on a formulation of the field equations as a single, first-order differential equation in a four-dimensional vector space. Similar formulations have been used by several authors as a basis for numerical field calculations [1], [2]. Vigants and Schlesinger [3] in a pioneering paper argue for this type of approach and point out the advantages obtained by treating the field equations as a set of first-order equations when use is made of existing mathematical techniques for

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